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B.Sc. Part III 5th Paper

Infinite Products (contd.)

Q. Consider the convergence of the infinite product

$$\left(1+\frac{x}{1^p}\right)\left(1+\frac{x}{2^p}\right)\left(1+\frac{x}{3^p}\right)\cdots\left(1+\frac{x}{n^p}\right)\cdots$$

where x is a natural number.

Soln

Here $\prod_{n=1}^{\infty}(1+U_n) = \left(1+\frac{x}{1^p}\right)\left(1+\frac{x}{2^p}\right)\cdots\left(1+\frac{x}{n^p}\right)\cdots$

$$\Rightarrow U_n = \frac{x}{n^p}$$

$$\Rightarrow \sum U_n = \sum \frac{x}{n^p} = x\sum \frac{1}{n^p}$$

$$= x\left(\frac{1}{1^p}+\frac{1}{2^p}+\cdots+\frac{1}{n^p}+\cdots\right)$$

The series $\sum U_n$ converges if $p>1$ and diverges if $p\leq 1$.

Hence the given product $\prod(1+U_n)$ converges if $p>1$ and diverges if $p\leq 1$.

Q. Prove that the product $\prod_{1}^{\infty}\left(1+\frac{1}{n^2}\right)$ is convergent

Soln. The given infinite product

$$= \left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots\left(1+\frac{1}{n^2}\right)\cdots$$

$$= \prod(1+U_n) = \prod\left(1+\frac{1}{n^2}\right)$$

$$\therefore \; U_n = \frac{1}{n^2}$$

$$\Rightarrow \sum U_n = \sum \frac{1}{n^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}+\cdots$$

$\because \sum \frac{1}{n^p}$ is convergent if $p>1$

$\Rightarrow \sum \frac{1}{n^2}$ is convergent as $p=2$ here.

$\Rightarrow \prod\left(1+\frac{1}{n^2}\right)$ is also convergent.

$\boxed{\text{$\prod(1+a_n)$ converges to a finite non-zero limit or diverges to ∞ according as the series $\sum a_n$ is convergent or divergent.}}$